Carleton UNIVERSITY
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Mathematics Tutorial Series
Differential Calculus #15

Inverse trig functions

If $y = sin\, x$ then we might be given the value of y and then want the angle x.

We write this as $x = \sin^{-1} y$ or $x = \arcsin(y)$ or $x = \text{asin}(y)$.

It means that **x is the angle with sine equal to y.**

Switch x and y. We are going to use $\sin^{-1} x$ as a function of x.

$y = \sin^{-1} x = \arcsin x = \text{asin}\, x$ means that y is an angle with $\sin y = x$.

Inverse sine of $x = \sin^{-1} x$
Arcsine of $x = \arcsin x$.

Note: This is an overuse for the exponent -1.

$\sin^{-1} x$ has many uses and so we must get familiar with it.

Examples:

$$\sin^{-1}(0) = \arcsin(0) = 0$$
$$\sin^{-1}(1) = \arcsin(1) = \frac{\pi}{2}$$
$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

Domain and range

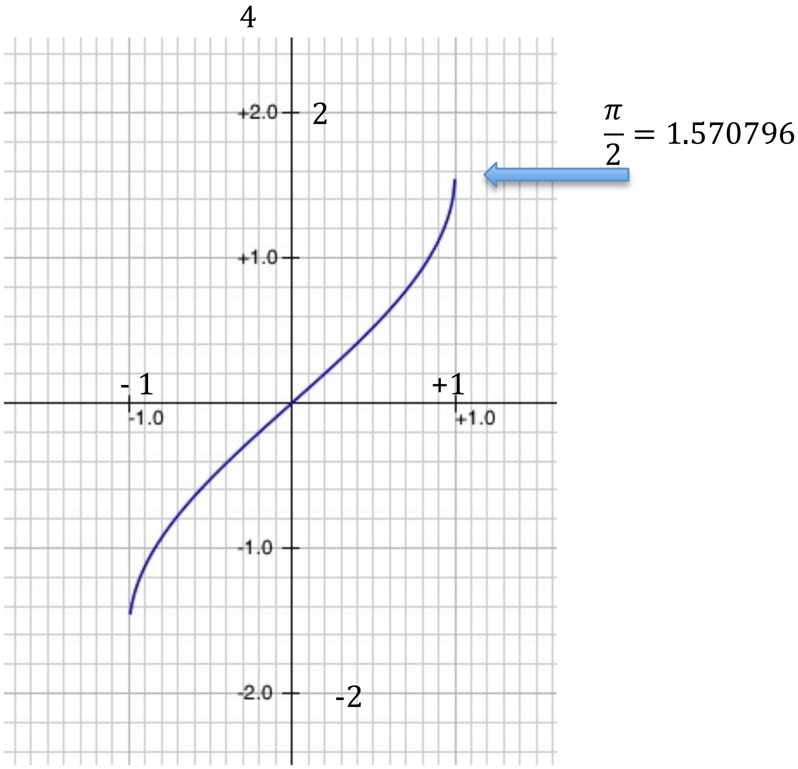
When we write $y = \arcsin x$ we are saying that the y is an angle with $sin\, y = x$.

Since the sine of an angle is between -1 and + 1, we can only evaluate $\arcsin(x)$ if $-1 \leq x \leq +1$. This is **the domain** of $\arcsin(x)$

To get a unique value for y we will have to specify the range to represent one period of sine. There are infinitely many choices – by convention we choose:

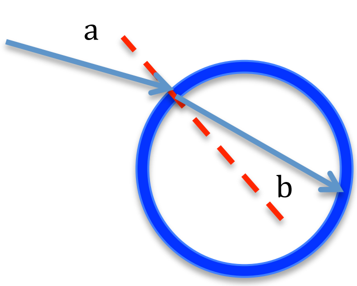
Range: $-\frac{\pi}{2} \leq y \leq +\frac{\pi}{2}$

Graph of $y = \sin^{-1} x$



Example:
Refraction

Refraction happens when sunlight hits a raindrop because light travels slightly slower in water than in air.



The angles a and b are not equal. They are related by:

$$\sin b = \frac{3}{4} \sin a$$

with ¾ being about right for visible wavelengths of light.

So

$$b = sin^{-1}(\frac{3}{4} \sin a)$$

$$b = \arcsin(\frac{3}{4} \sin a)$$

Derivative of Inverse Sine

If $y = \arcsin x$ then $x = \sin y$.

Differentiate implicitly.

$$x' = (\sin y)'$$

$$1 = \cos y \;\; y'$$

$$y' = \frac{1}{\cos y}$$

We want to change this to an expression in x.

From $\sin^2 y + \cos^2 y = 1$ we get

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \,.$$

So

$$(\arcsin x)' = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

The Other Inverse Trigonometric Functions

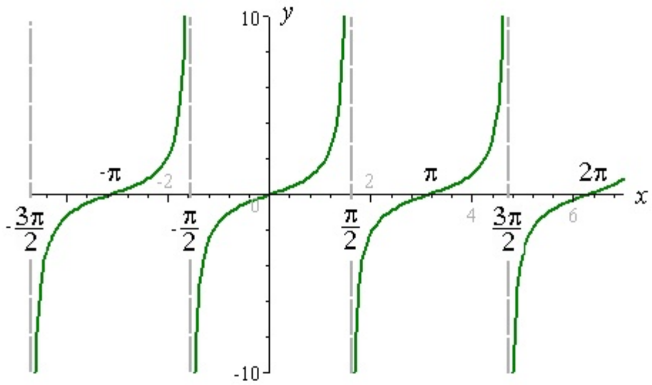
By restricting domain and range, we can define inverses for all the trig functions.

Only $\arcsin x$ and $\arctan x$ are commonly used.

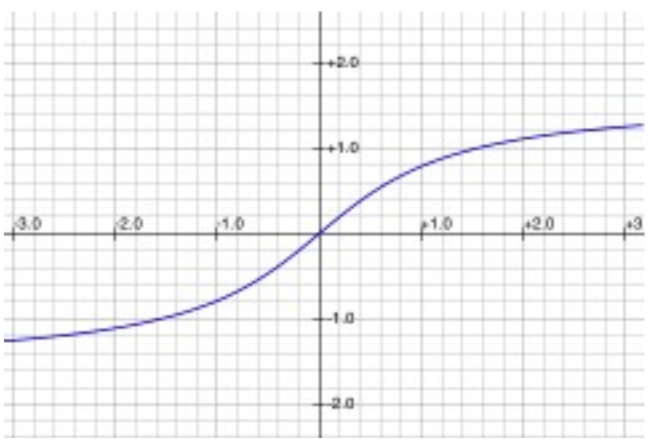
Inverse Tangent Function

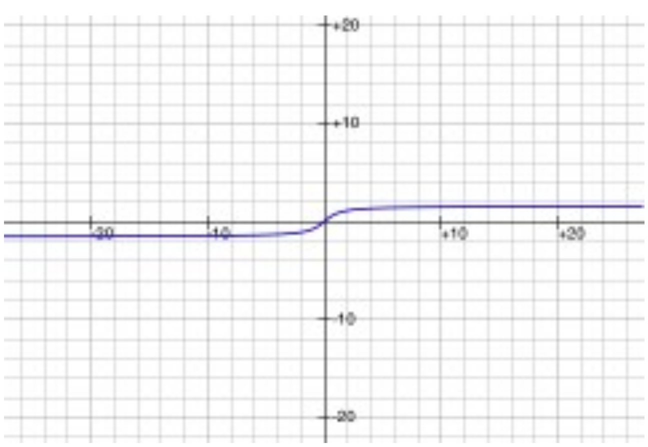
We write $y = \arctan x = \tan^{-1} x = \text{atan}\, x$ if $x = \tan y$.

The graph of the tangent looks like this:



So the graph of the inverse tangent looks like this:





Domain is all real numbers.
Range is $-\frac{\pi}{2} < y < +\frac{\pi}{2}$

Derivative of Inverse Tangent

If the basic trig identity: $\sin^2 y + \cos^2 y = 1$ is divided by $\cos^2 y$, we get

$$\tan^2 y + 1 = \sec^2 y$$

If $y = \tan^{-1} x$ then $x = \tan y$ and we differentiate implicitly:

$$1 = \sec^2 y \; y'$$

So

$$y' = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

This is important because it gives us a function with derivative $\frac{1}{1+x^2}$.

Function Menu on a Calculator

This shows where the inverse trigs are placed.
They are labeled as “asin”, “acos” and “atan”

Summary

Function	$\mathbf{sin^{-1}x}$	$\mathbf{tan^{-1}x}$
Domain	$-1 \leq x \leq +1$	All real numbers
Range	$-\frac{\pi}{2} \leq x \leq +\frac{\pi}{2}$	$-1 < x < +1$
Derivative	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$

$\sin^{-1} x = \arcsin x = \text{asin}\, x$

$\tan^{-1} x = \arctan x = \text{atan}\, x$